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Measuring CP-violating phases through studying the polarization of the final particles in $\mu \to eee$

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ABSTRACT

It is shown that the polarizations of the final particles in $\mu^+ \to e^+e^-e^+$ provide us with information on the CP-violating phases of the effective Lagrangian leading to this Lepton Flavor Violating (LFV) decay.

1. Introduction

In the context of the "old" Standard Model (SM) with zero neutrino masses, the lepton flavor is conserved. As a result, the LFV decays such as $\mu \to eee$ and $\mu \to e\gamma$ are strictly forbidden within the "old" SM. Within the "new" SM with sources of LFV in the neutrino mass matrix, such decays are in principle allowed but their rates are suppressed by powers of the neutrino mass and are therefore beyond the reach of any searches in the foreseeable future [1]. However, a variety of beyond SM scenarios can lead to rates for these processes exceeding the present experimental bounds [2]:

$$\mathrm{Br}(\mu \to e\gamma) < 1.2 \times 10^{-11} \quad \text{and} \quad \mathrm{Br}(\mu^- \to e^-e^+e^-) < 1.0 \times 10^{-12}. \tag{1}$$

Notice that the bound on $\mu \to eee$ is even stronger than the famous bound on $\mu \to e\gamma$. In the context of models like R-parity conserving MSSM in which the new particles can appear only in even numbers in each vertex, these processes can only take place at the loop level. Since $\mu \to eee$ is a three body decay, in such a model, $\mathrm{Br}(\mu \to eee)$ is suppressed relative to $\mathrm{Br}(\mu \to e\gamma)$ by a factor typically of order of $e^2/(16\pi^2)\log(m_\mu/m_e)$ [3]. However, in the models that new particles can appear in odd numbers at each vertex (*e.g.*, in R-parity violating MSSM) the process $\mu \to eee$ can take place at tree level and as a result, its rate can even exceed that of $\mu \to e\gamma$ [4].

It is rather well known that by measuring the angular distribution of the final particles relative to the spin of the initial muon in $\mu \to e\gamma$ and $\mu \to eee$, one can derive information on the chiral nature of the effective Lagrangian leading to this process [4–6]. In the case of $\mu \to eee$, as shown in the literature [4,7,8], the angular distribution of the final particles relative to the spin of the initial muon also yields information on certain combinations of the CP-violating phases.

Recently, it has been shown in [9–11] that if we measure the polarization of the emitted particles in $\mu \to e\gamma$ and $\mu N \to eN$, we can derive information on the CP-violating parameters of the theory. It was pointed out in [9,10] that by measuring the spin of the more energetic final positron in $\mu^+ \to e^+e^-e^+$, some information on the CP-violating phases can be derived. The analysis was performed in the framework of the models such as R-parity conserving MSSM, in which the dominant contribution to $\mu \to eee$ comes from a penguin diagram (i.e., $\mu^+ \to \gamma^* e^+ \to e^+e^-e^+$). In this Letter, we focus on the case that $\mu^+ \to e^+e^-e^+$ happens at the tree level through

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the exchange of heavy particles. We show that the transverse polarizations of the emitted particles in $\mu^+ \to e^+e^-e^+$ provide us with information on the CP-violating properties of the effective Lagrangian. The information derived from the spins of the emitted electrons and positrons are complementary to each other. In this Letter, we focus on $\mu^+ \to e^+e^-e^+$. Similar arguments hold for $\mu^- \to e^-e^+e^-$.

We show that the method proposed here is sensitive to a combination of the phases in the effective Lagrangian which is different from those that can be derived by methods discussed in the literature [4,7,8]. The effectiveness of each method depends on the relative magnitude of the different terms in the effective Lagrangian which in turn depends on the details of the underlying model.

2. The rate of $\mu^+ \to e^+e^-e^+$

Consider a general beyond SM scenario leading to $\mu^+ \to e^+e^-e^+$. After integrating out the heavy states, the effect can be described by an effective Lagrangian of form

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \tag{2}$$

where

$$\begin{aligned}\mathcal{L}_1 = {} & B_1\left(\bar{\mu}\frac{1+\gamma_5}{2}e\right)\left(\bar{e}\frac{1-\gamma_5}{2}e\right) + B_2\left(\bar{\mu}\frac{1-\gamma_5}{2}e\right)\left(\bar{e}\frac{1+\gamma_5}{2}e\right) + C_1\left(\bar{\mu}\frac{1+\gamma_5}{2}e\right)\left(\bar{e}\frac{1+\gamma_5}{2}e\right) \\ & + C_2\left(\bar{\mu}\frac{1-\gamma_5}{2}e\right)\left(\bar{e}\frac{1-\gamma_5}{2}e\right) + G_1\left(\bar{\mu}\gamma^\nu\frac{1+\gamma_5}{2}e\right)\left(\bar{e}\gamma_\nu\frac{1+\gamma_5}{2}e\right) + G_2\left(\bar{\mu}\gamma^\nu\frac{1-\gamma_5}{2}e\right)\left(\bar{e}\gamma_\nu\frac{1-\gamma_5}{2}e\right) + \text{H.c.}\end{aligned} \tag{3}$$

and

$$\mathcal{L}_2 = A_L\bar{\mu}\left[\gamma^\mu,\gamma^\nu\right]\frac{1+\gamma_5}{2}eF_{\mu\nu} + A_R\bar{\mu}\left[\gamma^\mu,\gamma^\nu\right]\frac{1-\gamma_5}{2}eF_{\mu\nu} + \text{H.c.} \tag{4}$$

Notice that by using the identities $(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} \equiv 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$ and $(\bar{\sigma}^\mu)_{\alpha\beta} = \epsilon_{\beta\delta}(\sigma^\mu)_{\delta\gamma}\epsilon_{\gamma\alpha}$ (where $\epsilon_{11} = \epsilon_{00} = 0$ and $\epsilon_{10} = -\epsilon_{01} = 1$) and employing the fact that the fermions anti-commute, we can rewrite the terms on the first line of Eq. (3) as

$$-\frac{B_1}{2}\left(\bar{\mu}\frac{1+\gamma_5}{2}\gamma^\nu e\right)\left(\bar{e}\frac{1-\gamma_5}{2}\gamma_\nu e\right) - \frac{B_2}{2}\left(\bar{\mu}\frac{1-\gamma_5}{2}\gamma^\nu e\right)\left(\bar{e}\frac{1+\gamma_5}{2}\gamma_\nu e\right).$$

In the literature, it has been shown that by studying the angular distribution of the final particles relative to the spin of the initial muon, one can derive information on $\mathrm{Re}[A_R B_1^* - A_L B_2^*]$, $\mathrm{Re}[A_R G_2^* - A_L G_1^*]$, $\mathrm{Im}[A_R B_1^* + A_L B_2^*]$ and $\mathrm{Im}[A_R G_2^* + A_L G_1^*]$ (see, *e.g.*, [4,7,8]). However, by this method the phases of C_1 and C_2 cannot be derived. Moreover, if A_L and A_R are zero or suppressed, this method loses its effectiveness.

By studying the energy spectrum of the emitted particles, it will be possible to differentiate between the different terms in $\mathcal{L}_1$ and $\mathcal{L}_2$. For example, the A_L and A_R couplings lead to a sharp peak in the distribution of the square of the invariant mass of a e^-e^+ pair [*i.e.*, in the distribution of $(P_{e^-} - P_{e^+})^2$] at $(m_\mu/2)^2$. Such a peak does not appear if the dominant contribution comes from Eq. (3). The A_L and A_R couplings are generally loop suppressed but, as we see below, terms in $\mathcal{L}_1$ can appear in a wide range of models at tree level. In the present Letter, we only consider terms in $\mathcal{L}_1$.

The B_i and C_i couplings in $\mathcal{L}_1$ can originate from the exchange of a heavy neutral scalar field (or fields). Let us demonstrate this through a simple toy model. Consider two heavy *complex* fields, ϕ_1 and ϕ_2 with the following couplings

$$g_{\mu L}\phi_1\bar{\mu}\frac{1+\gamma_5}{2}e + g_{\mu R}\phi_2\bar{\mu}\frac{1-\gamma_5}{2}e + g_{eL}\phi_1\bar{e}\frac{1+\gamma_5}{2}e + g_{eR}\phi_2\bar{e}\frac{1-\gamma_5}{2}e + \text{H.c.}$$

It is straightforward to show that the effective Lagrangian resulting from integrating out the heavy states ϕ_1 and ϕ_2 is of form (3) with

$$B_1 = \frac{g_{\mu L}g_{eL}^*}{m_{\phi_1}^2}, \qquad B_2 = \frac{g_{\mu R}g_{eR}^*}{m_{\phi_2}^2}, \qquad C_1 = C_2 = 0.$$

If we swap ϕ_1 and ϕ_2 in the third and fourth terms (*i.e.*, taking $\mathcal{L} = [g_{\mu L}\phi_1\bar{\mu}(1+\gamma_5)e + g_{\mu R}\phi_2\bar{\mu}(1-\gamma_5)e + g_{eL}\phi_2\bar{e}(1+\gamma_5)e + g_{eR}\phi_1\bar{e}(1-\gamma_5)e + \text{H.c.}]/2$), we find $C_1 = g_{\mu L}g_{eR}^*/m_{\phi_1}^2$, $C_2 = g_{\mu R}g_{eL}^*/m_{\phi_2}^2$ and $B_1 = B_2 = 0$. Taking ϕ_1 and ϕ_2 *real*, we find that B_1, B_2, C_1 and C_2 are all nonzero. It can similarly be shown that G_1 and G_2 can originate from the exchange of a doubly charged scalar field. The R-parity violating MSSM at the tree level leads to couplings of form B_1 and B_2 [4]. That is while in this model, the A_L and A_R couplings are loop suppressed.

Notice that while the B_i couplings are chirality conserving, the C_i and G_i couplings are chirality-flipping. Notice that under the parity transformation, B_i and C_i transform as $B_1 \Leftrightarrow B_2$, $G_1 \Leftrightarrow G_2$ and $C_1 \Leftrightarrow C_2$. Thus, $|B_1 - B_2|$, $|G_1 - G_2|$ and $|C_1 - C_2|$ can be considered as measures of the parity violation. Under the CP transformation

$$B_1 \Rightarrow \eta B_1^*, \qquad B_2 \Rightarrow \eta B_2^*, \qquad G_1 \Rightarrow \eta G_1^*, \qquad G_2 \Rightarrow \eta G_2^*, \qquad C_1 \Rightarrow \eta C_1^* \quad \text{and} \quad C_2 \Rightarrow \eta C_2^*,$$

where η is a pure phase that comes from the freedom in the definition of the CP-conjugate of the electron and the muon. By rephasing the electron and/or the muon field, either of these couplings can be made real. Thus, the Lagrangian in Eq. (3) contains five physical CP-violating phases.

From the formulas derived Appendix A, we find that the differential decay rate of $\mu^+ \to e^+e^-e^+$ is

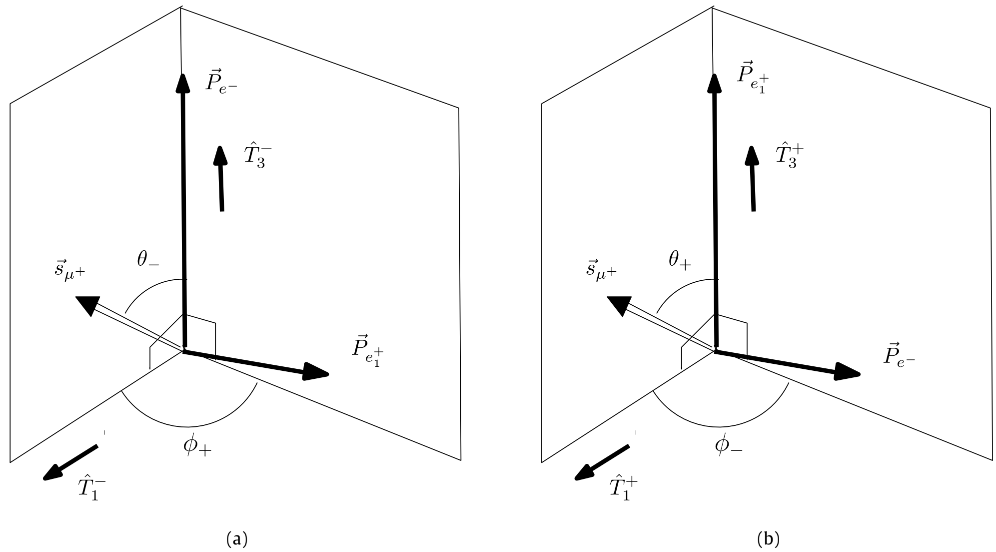

Fig. 1. These figures schematically depict the direction of the momenta of the final particles in the LFV decay $\mu^+ \to e_1^+ e^- e_2^+$ relative to the spin of the anti-muon in its rest frame. Both figures correspond to a single decay (a) illustrating $\hat{T}^-$, θ_- and ϕ_+ and (b) illustrating $\hat{T}^+$, θ_+ and ϕ_-.

$$\int_0^{2\pi} \sum_{\vec{s}_{e^-},\vec{s}_{e_1^+}\vec{s}_{e_2^+}} \frac{d^3\Gamma(\mu^+ \to e_1^+ e^- e_2^+)}{dE_e\, dE_{e_1^+}\, d\Omega} d\phi_+$$
$$= \frac{1}{128\pi^4}\Big[\big(|B_1|^2+|B_2|^2\big)\big(-m_\mu^3+4m_\mu^2 E_{e_1^+}+3m_\mu^2 E_e-2E_e^2 m_\mu-4m_\mu E_{e_1^+}^2-4m_\mu E_e E_{e_1^+}\big)$$
$$+\big(|B_1|^2-|B_2|^2\big)\mathbb{P}_\mu\cos\theta_-\big(m_\mu^4-4m_\mu^3 E_{e_1^+}-3m_\mu^3 E_e+8m_\mu^2 E_e E_{e_1^+}+4m_\mu^2 E_{e_1^+}^2+4m_\mu^2 E_e^2-2m_\mu E_e^3-6m_\mu E_e^2 E_{e_1^+}\big)/E_e$$
$$+\big[\big(|C_1|^2+16|G_2|^2\big)(1+\mathbb{P}_\mu\cos\theta_-)+\big(|C_2|^2+16|G_1|^2\big)(1-\mathbb{P}_\mu\cos\theta_-)\big]E_e m_\mu(m_\mu-2E_e)\Big], \tag{5}$$

where $d\Omega$ is the differential solid angle determining the orientation of the emitted electron. We have summed over the spins of the final particles and integrated over ϕ_+ which determines the azimuthal angle of the emitted positrons (see Fig. 1).

Integrating over the energies of the final particles, we find

$$\sum_{\vec{s}_{e^-},\vec{s}_{e_1^+},\vec{s}_{e_2^+}} \frac{d\Gamma}{d\cos\Omega} = \frac{0.0208}{128\pi^4} m_\mu^5\Bigg[\bigg(|B_1|^2+|B_2|^2+\frac{|C_1|^2+|C_2|^2}{2}+8|G_1|^2+8|G_2|^2\bigg)$$
$$+\mathbb{P}_\mu\cos\theta_-\bigg(\frac{|B_2|^2-|B_1|^2}{3}+\frac{|C_1|^2-|C_2|^2}{2}+8\big(|G_2|^2-|G_1|^2\big)\bigg)\Bigg]. \tag{6}$$

The total rate of $\int(d\Gamma/d\Omega)d\Omega$ is given by $(|B_1|^2+|B_2|^2+(|C_1|^2+|C_2|^2)/2)+8(|G_1|^2+|G_2|^2)$. From the bound on $\mathrm{Br}(\mu\to eee)$ (see Eq. (1)), we find

$$|B_1|^2+|B_2|^2+\frac{|C_1|^2+|C_2|^2}{2}+8\big(|G_1|^2+|G_2|^2\big)<\frac{1}{(200\ \mathrm{TeV})^4}.$$

From Eq. (5), we observe that the dependence of the coefficients of $|B_1|^2+|B_2|^2$ and $|C_1|^2+|C_2|^2+16(|G_1|^2+|G_2|^2)$ on $E_{e_1^+}$ and E_e are different. As a result, by studying the energy spectrum of the final particles, it will be possible to extract $|B_1|^2+|B_2|^2$ and $|C_1|^2+|C_2|^2+16(|G_1|^2+|G_2|^2)$. If in addition to the energy spectrum, the angular distribution of the electron relative to $\vec{s}_\mu$ is measured, it will be possible to extract the parity violating combinations $|B_1|^2-|B_2|^2$ and $|C_1|^2-|C_2|^2+16(|G_2|^2-|G_1|^2)$. Of course, the larger the polarization of the initial muon, the higher the sensitivity of the angular distribution to these combinations. Thus, in principle by measuring the spectrum of the emitted particles and angular distribution of the final electron, it will be possible to derive the absolute values of the couplings. The angular distribution of the final positrons also give information on $|B_1|^2-|B_2|^2$ and $|C_1|^2-|C_2|^2+16(|G_2|^2-|G_1|^2)$ (see Appendix A). Thus, the following combinations can be measured from the study of the angular distribution plus energy spectrum measurements:

$$|B_1|^2,\quad |B_2|^2,\quad |C_1|^2+16|G_2|^2\quad\text{and}\quad |C_2|^2+16|G_1|^2. \tag{7}$$

However, without sensitivity to the spins of the final particles, it is not possible to derive information on the phases of the couplings. In Section 2.1, we explore what can be learned from the polarization of the emitted electron. In Section 2.2, we discuss the polarization of the positron.

2.1. Polarization of the emitted electron

Consider the decay $\mu^+ \to e^+e^-e^+$ in the rest frame of the muon, where the final electron makes an angle of θ_- with the spin of the initial muon. Let us take the $\hat{T}_3^-$ direction parallel to the momentum of the emitted electron and $\hat{T}_2^- \equiv \hat{T}_3^- \times \vec{s}_\mu / |\hat{T}_3^- \times \vec{s}_\mu|$ (see Fig. 1). The spin of the electron is determined by $(c_{e^-}\ d_{e^-})$ with $(|c_{e^-}|^2 + |d_{e^-}|^2)^{1/2} = 1$. That is the components of the spin of the electron are

$$\hat{T}_3^- \cdot \vec{s}_{e^-} = |c_{e^-}|^2 - |d_{e^-}|^2, \qquad \hat{T}_1^- \cdot \vec{s}_{e^-} = 2\,\mathrm{Re}\big[c_{e^-}^* d_{e^-}\big] \quad \text{and} \quad \hat{T}_2^- \cdot \vec{s}_{e^-} = 2\,\mathrm{Im}\big[c_{e^-}^* d_{e^-}\big]. \tag{8}$$

From the formulas in Appendix A, we find that the differential decay rate in the rest frame of the muon is

$$\begin{aligned}
&\sum_{\vec{s}_{e_1^+}, \vec{s}_{e_2^+}} \frac{d\Gamma(\mu^+ \to e_1^+ e^- e_2^+)}{d\cos\Omega} \\
&= \int_0^{2\pi} \int_0^{m_\mu/2} \int_{m_\mu/2 - E_e}^{m_\mu/2} \sum_{\vec{s}_{e_1^+}, \vec{s}_{e_2^+}} |M|^2 E_{e_1^+} E_e (m_\mu - E_{e_1^+} - E_e)\, dE_{e_1^+}\, dE_e\, d\phi_+ \\
&= \frac{m_\mu^5}{128\pi^4}\Big[(0.0208)\big[|B_1|^2|c_{e^-}|^2 + |B_2|^2|d_{e^-}|^2 + \big(|C_1|^2/2 + 8|G_2|^2\big)|d_{e^-}|^2 + \big(|C_2|^2/2 + 8|G_1|^2\big)|c_{e^-}|^2\big] \\
&\quad + (0.0208)\mathbb{P}_\mu \cos\theta_- \big[|B_2|^2|d_{e^-}|^2/3 - |B_1|^2|c_{e^-}|^2/3 + \big(|C_1|^2/2 + 8|G_2|^2\big)|d_{e^-}|^2 - \big(|C_2|^2/2 + 8|G_1|^2\big)|c_{e^-}|^2\big] \\
&\quad - (0.0832)\mathbb{P}_\mu \sin\theta_- \big(\mathrm{Re}\big[G_1 C_1^* d_{e^-} c_{e^-}^*\big] + \mathrm{Re}\big[G_2 C_2^* d_{e^-}^* c_{e^-}\big]\big) + 0.055\,\mathrm{Re}\big[B_1 B_2^* d_{e^-} c_{e^-}^*\big]\mathbb{P}_\mu \sin\theta_-\Big],
\end{aligned} \tag{9}$$

where $\mathbb{P}_\mu$ is the polarization of the initial muon and $d\Omega$ represents the differential solid angle of the momentum of the electron relative to the spin of the muon. To obtain this equation, we have summed over the spins of the emitted positrons and integrated over the azimuthal angle that the plane containing the momenta of these positrons makes with the plane of the spin of the anti-muon and the momentum of the electron. See Appendix A for the details. Notice that there is no interference term between the chirality-flipping and chirality-conserving couplings.

From Eq. (24) in Appendix A, we find that the longitudinal polarization of the electron is

$$\langle s_{T_3^-} \rangle \equiv \frac{d\Gamma/d\Omega|_{c_{e^-}=1,\, d_{e^-}=0} - d\Gamma/d\Omega|_{c_{e^-}=0,\, d_{e^-}=1}}{d\Gamma/d\Omega|_{c_{e^-}=1,\, d_{e^-}=0} + d\Gamma/d\Omega|_{c_{e^-}=0,\, d_{e^-}=1}} = \frac{P_1^- - \mathbb{P}_\mu \cos\theta_-\, P_2^-}{P_3^- + \mathbb{P}_\mu \cos\theta_-\, P_4^-}, \tag{10}$$

where

$$\begin{aligned}
P_1^- &= |B_1|^2 - |B_2|^2 - \frac{|C_1|^2}{2} + \frac{|C_2|^2}{2} + 8|G_1|^2 - 8|G_2|^2, \\
P_2^- &= \frac{|B_1|^2}{3} + \frac{|B_2|^2}{3} + \frac{|C_1|^2}{2} + \frac{|C_2|^2}{2} + 8|G_1|^2 + 8|G_2|^2, \\
P_3^- &= |B_1|^2 + |B_2|^2 + \frac{|C_1|^2}{2} + \frac{|C_2|^2}{2} + 8|G_1|^2 + 8|G_2|^2, \\
P_4^- &= -\frac{|B_1|^2}{3} + \frac{|B_2|^2}{3} + \frac{|C_1|^2}{2} - \frac{|C_2|^2}{2} - 8|G_1|^2 + 8|G_2|^2.
\end{aligned} \tag{11}$$

$\langle s_{T_3^-} \rangle$ is sensitive only to the absolute values of the couplings. Notice that by measuring $\langle s_{T_3^-} \rangle$ and its angular distribution, one can derive the same combinations that are listed in Eq. (7), *i.e.*, the combinations that can be extracted from the angular distribution and energy spectrum (without measuring the spin of the final particles). Derivation of these combinations from the measurement of $\langle s_{T_3^-} \rangle$ can be considered as a cross-check for the derivation from the energy spectrum. Notice that even in the $\mathbb{P}_\mu = 0$ limit, $\langle s_{T_3^-} \rangle$ is still non-vanishing and provides us with information on the parity violating combination P_1. That is while in this limit, the angular distribution of the electron is uniform and has no sensitivity to parity violation in the effective Lagrangian (3) (see Eq. (5)).

The transverse polarizations are

$$\langle s_{T_1^-} \rangle \equiv \frac{d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=1/\sqrt{2}} - d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=-1/\sqrt{2}}}{d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=1/\sqrt{2}} + d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=-1/\sqrt{2}}} = \frac{\mathrm{Re}[(2.6)B_1^* B_2 - 4G_1^* C_1 - 4G_2 C_2^*]\sin\theta_- \mathbb{P}_\mu}{P_3^- + P_4^- \mathbb{P}_\mu \cos\theta_-} \tag{12}$$

and

$$\langle s_{T_2^-} \rangle \equiv \frac{d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=i/\sqrt{2}} - d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=-i/\sqrt{2}}}{d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=i/\sqrt{2}} + d\Gamma/d\Omega|_{c_{e^-}=1/\sqrt{2},\, d_{e^-}=-i/\sqrt{2}}} = \frac{\mathrm{Im}[(2.6)B_1^* B_2 - 4G_1^* C_1 - 4G_2 C_2^*]\sin\theta_- \mathbb{P}_\mu}{P_3^- + P_4^- \mathbb{P}_\mu \cos\theta_-}. \tag{13}$$

The transverse polarizations are maximal for electrons emitted in the direction perpendicular to the spin of the muon: *i.e.*, $\theta_- = \pi/2$. Notice that $\langle s_{T_1^-} \rangle$ and $\langle s_{T_2^-} \rangle$ provide independent information on the real and imaginary parts of the same combination; *i.e.*, $(2.6B_1^* B_2 - 4G_1^* C_1 - 4G_2 C_2^*)$. Notice that both the moduli and the phase of this combination contain extra information beyond the combinations listed in Eq. (7). In the context of various models G_i and C_i can vanish. For example, as explained in the previous section, if the effective

Lagrangian comes from integrating out neutral scalars, the G_i couplings will vanish. Within such models, $\langle s_{T_1^-}\rangle \propto \mathrm{Re}[B_1 B_2^*]$ and $\langle s_{T_2^-}\rangle \propto \mathrm{Im}[B_1 B_2^*]$. Considering that $|B_i|$ can be independently measured, the derivation of $\arg[B_1 B_2^*]$ from both $\langle s_{T_1^-}\rangle$ and $\langle s_{T_2^-}\rangle$ can be considered as a cross-check.

In the limit of unpolarized muon (*i.e.*, $\mathbb{P}_\mu \to 0$), $\langle s_{T_1^-}\rangle$ and $\langle s_{T_2^-}\rangle$ vanish. This is understandable because in the $\mathbb{P}_\mu = 0$ limit, there is no preferred directions so we cannot define $\hat{T}_1^-$ and $\hat{T}_2^-$ (see Fig. 1). Thus, to derive the CP-violating phase, $\arg[B_1 B_2^*]$, it is necessary to have a source of polarized muon which is quite feasible. For example, if the muons are produced by the decay of pions at rest, they will be almost 100% polarized. In fact, this is the case for the on-going MEG experiment.

By measuring the polarization of the emitted electron, it is not possible to derive the values of all the CP-violating phases of the Lagrangian (3). Measurement of the angular distribution of the positrons does not provide any further information. As we shall see next, the transverse polarizations of the emitted positrons provide complementary information on the phases.

2.2. Polarization of the positron

Consider the decay $\mu^+ \to e_1^+ e^- e_2^+$ in the rest frame of the muon where one of the final positrons makes an angle of θ_+ with the spin of the initial muon. Let us take the $\hat{T}_3^+$ direction parallel to the momentum of e_1^+ and $\hat{T}_2^+ \equiv \hat{T}_3^+ \times \vec{s}_\mu / |\hat{T}_3^+ \times \vec{s}_\mu|$ (see Fig. 1). The spin of e_1^+ is determined by $(c_{e_1^+} d_{e_1^+})$ with $(|c_{e_1^+}|^2 + |d_{e_1^+}|^2)^{1/2} = 1$. The components of the spin are

$$\hat{T}_3^+ \cdot \vec{s}_{e_1^+} = |c_{e_1^+}|^2 - |d_{e_1^+}|^2, \qquad \hat{T}_1^+ \cdot \vec{s}_{e_1^+} = 2\,\mathrm{Re}\big[c_{e_1^+}^* d_{e_1^+}\big] \quad \text{and} \quad \hat{T}_2^+ \cdot \vec{s}_{e_1^+} = 2\,\mathrm{Im}\big[c_{e_1^+}^* d_{e_1^+}\big]. \tag{14}$$

From the formula in Appendix A, we find that the differential decay rate in the rest frame of the muon is

$$\begin{aligned}
&\sum_{\vec{s}_{e^-}, \vec{s}_{e_2^+}} \frac{d\Gamma(\mu^+ \to e_1^+ e^- e_2^+)}{d\cos\Omega} \\
&= \int_0^{2\pi} \int_0^{m_\mu/2} \int_{m_\mu/2 - E_e}^{m_\mu/2} \sum_{\vec{s}_{e^-}, \vec{s}_{e_2^+}} |M|^2 E_{e_1^+} E_e (m_\mu - E_{e_1^+} - E_e)\, dE_e\, dE_{e_1^+}\, d\phi_- \\
&= (0.0104) \frac{m_\mu^5}{128\pi^4} \bigg[\big(|C_1|^2 |c_{e_1^+}|^2 + |C_2|^2 |d_{e_1^+}|^2\big) - \big(|C_1|^2 |c_{e_1^+}|^2 - |C_2|^2 |d_{e_1^+}|^2\big) \frac{\mathbb{P}_\mu \cos\theta_+}{3} \\
&\quad + |B_1|^2 + |B_2|^2 + \big[|B_1|^2 \big(|c_{e_1^+}|^2 - |d_{e_1^+}|^2/3\big) - |B_2|^2 \big(|d_{e_1^+}|^2 - |c_{e_1^+}|^2/3\big)\big] \mathbb{P}_\mu \cos\theta_+ \\
&\quad + 16\big(|G_2|^2 |c_{e_1^+}|^2 + |G_1|^2 |d_{e_1^+}|^2\big) - \frac{16}{3}\big(|G_2|^2 |c_{e_1^+}|^2 - |G_1|^2 |d_{e_1^+}|^2\big) \mathbb{P}_\mu \cos\theta_+ \\
&\quad + \big(\mathrm{Re}\big[B_1 C_2^* c_{e_1^+} d_{e_1^+}^*\big] + \mathrm{Re}\big[B_2 C_1^* c_{e_1^+}^* d_{e_1^+}\big]\big) \mathbb{P}_\mu \sin\theta_+ + 24\big(\mathrm{Re}\big[G_1 B_1^* c_{e_1^+}^* d_{e_1^+}\big] - \mathrm{Re}\big[G_2 B_2^* c_{e_1^+} d_{e_1^+}^*\big]\big) \mathbb{P}_\mu \sin\theta_+ \bigg]
\end{aligned} \tag{15}$$

where $d\Omega$ represents the differential solid angle of the momentum of e_1^+ relative to the spin of the muon.

From the above equation, we find that the longitudinal polarization of e_1^+ is

$$\langle s_{T_3^+}\rangle = \frac{d\Gamma/d\Omega|_{c_{e_1^+}=1,\, d_{e_1^+}=0} - d\Gamma/d\Omega|_{c_{e_1^+}=0,\, d_{e_1^+}=1}}{d\Gamma/d\Omega|_{c_{e_1^+}=1,\, d_{e_1^+}=0} + d\Gamma/d\Omega|_{c_{e_1^+}=0,\, d_{e_1^+}=1}} = \frac{P_1^+ + P_2^+ \mathbb{P}_\mu \cos\theta_+/3}{P_3^+ + P_4^+ \mathbb{P}_\mu \cos\theta_+/3}, \tag{16}$$

where

$$\begin{aligned}
P_1^+ &= |C_1|^2 - |C_2|^2 + 16|G_2|^2 - 16|G_1|^2, \\
P_2^+ &= -|C_2|^2 - |C_1|^2 - 16|G_2|^2 - 16|G_1|^2 + 4|B_1|^2 + 4|B_2|^2, \\
P_3^+ &= |C_1|^2 + |C_2|^2 + 16\big(|G_1|^2 + |G_2|^2\big) + 2\big(|B_1|^2 + |B_2|^2\big), \\
P_4^+ &= |C_2|^2 - |C_1|^2 - 16|G_2|^2 + 16|G_1|^2 + 2|B_1|^2 - 2|B_2|^2.
\end{aligned} \tag{17}$$

The transverse polarizations are

$$\langle s_{T_1^+}\rangle = \frac{d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=1/\sqrt{2}} - d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e^-}=-1/\sqrt{2}}}{d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=1/\sqrt{2}} + d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=-1/\sqrt{2}}} = \frac{\mathrm{Re}[B_1 C_2^* + B_2^* C_1 + 24 G_1^* B_1 - 24 G_2 B_2^*] \sin\theta_+ \mathbb{P}_\mu}{P_3^+ + P_4^+ \mathbb{P}_\mu \cos\theta_+/3} \tag{18}$$

and

$$\langle s_{T_2^+}\rangle = \frac{d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=i/\sqrt{2}} - d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=-i/\sqrt{2}}}{d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=i/\sqrt{2}} + d\Gamma/d\Omega|_{c_{e_1^+}=1/\sqrt{2},\, d_{e_1^+}=-i/\sqrt{2}}} = \frac{\mathrm{Im}[B_1 C_2^* + B_2^* C_1 + 24 G_1^* B_1 - 24 G_2 B_2^*] \sin\theta_+ \mathbb{P}_\mu}{P_3^+ + P_4^+ \mathbb{P}_\mu \cos\theta_+/3}. \tag{19}$$

Like the case of the electron discussed in Section 2.1, the longitudinal polarization, $\langle s_{T_3^+}\rangle$, does not give information on the CP-violating phases and can be used only as a cross-check for the derivation of the combinations listed in Eq. (7) by the methods discussed earlier. The ratio of $\langle s_{T_1^+}\rangle$ and $\langle s_{T_2^+}\rangle$ gives $\arg[B_1C_2^* + B_2^*C_1 + 24G_1^*B_1 - 24G_2B_2^*]$. Considering that the absolute value of this combination is also unknown, $|\langle s_{T_2^+}\rangle|^2 + |\langle s_{T_1^+}\rangle|^2$ provides an independent piece of information. We have integrated over ϕ_- which means the measurement of the direction of the emitted electron is not necessary for this analysis.

3. Conclusions and prospects

A large variety of the beyond standard models predict a sizeable rate for $\mu^+ \to e^+e^-e^+$ exceeding the present experimental bound. In principle, by studying the energy spectrum of the final particles and their angular distribution, it is possible to derive the form of the terms in the effective Lagrangian leading to this process and extract information on the *absolute values* of the couplings (see Eq. (7)). The effective Lagrangian responsible for $\mu^+ \to e^+e^-e^+$ can include new CP-violating phases. In order to derive information on the CP-violating phases, we have suggested to measure the polarization of the emitted particles. In this Letter, we have focused on the effective Lagrangian in Eq. (3) that can result from integrating out heavy scalar fields with LFV couplings at the tree level. The rest of the terms (*i.e.*, A_L and A_R) are expected to be loop suppressed and are neglected in this analysis. We have shown that the transverse polarization of the emitted electron in $\mu^+ \to e^+e^-e^+$ is sensitive to $\arg[2.6B_1B_2^* - 4G_1^*C_1 - 4G_2C_2^*]$ [see Eqs. (12), (13)]. That is while the transverse polarizations of the emitted positron is given by $\arg[B_1C_2^* + B_2^*C_1 + 24G_1^*B_1 - 24G_2B_2^*]$. From Eqs. (12), (13), (18), (19), we observe that if the initial muon is unpolarized (*i.e.*, $\mathbb{P}_\mu = 0$) the transverse polarizations of the emitted particles vanish. Thus, in order to derive the CP-violating phases, a source of polarized muons is required.

In sum, the effective Lagrangian in Eq. (3) includes six new couplings and five physical phases. By measuring the energy spectrum of the final particles and the angular distributions relative to the spin of the initial muon, one can derive the CP-conserving combinations listed in Eq. (7): *i.e.*, four out of the six CP-conserving quantities. Neglecting the loop suppressed A_L and A_R couplings, the angular distribution cannot provide information on the phases. The transverse polarizations of the emitted particles provide four independent pieces of information on the phases and couplings. This information is not enough to reconstruct all the couplings but considerably reduces the degeneracy in the parameter space.

We have also discussed the longitudinal polarization of the emitted particles. The longitudinal polarizations do not depend on the phases of the effective couplings. It is noteworthy that even in the $\mathbb{P}_\mu = 0$ limit, the longitudinal polarizations of the positron, $\langle s_{T_3^+}\rangle$, and the electron, $\langle s_{T_3^-}\rangle$, are nonzero and respectively yield information on the parity violating combinations $|C_1|^2 - |C_2|^2 + 16|G_2|^2 - 16|G_1|^2$ and $|C_1|^2 - |C_2|^2 + 16|G_2|^2 - 16|G_1|^2 + 2(|B_2|^2 - |B_1|^2)$. Remember that in the $\mathbb{P}_\mu = 0$ limit, there is no preferred direction so the angular distribution of the final particles is uniform and does not yield information on the parity-violating combinations.

There are running and/or under construction experiments that aim to probing signals for $\mu \to e\gamma$ [12] and $\mu - e$ conversion on nuclei [13] several orders of magnitudes below the present bounds on their rates. However, as shown in [4], it is possible that while $\mu^+ \to e^+e^-e^+$ is round the corner, the rates of $\mu \to e\gamma$ and $\mu - e$ conversion are too low to be probed. In fact as shown in [4], the three experiments provide us with complementary information on the parameters of the effective LFV Lagrangian. If the muons are produced from the decay of pions at rest (like the case of the running MEG experiment [12]), the initial muons in $\mu \to eee$ will be polarized. On the other hand, there are well-established techniques to measure the polarization of the emitted particles in this energy range. In fact, such polarimetry has been used to measure the Michel parameters since 80s [14]. As a result, if the rate of $\mu^+ \to e^+e^-e^+$ is close to the present bound and a hypothetical experiment finds statistically large number of such a process, performing the analysis proposed in this Letter sounds possible.

In this Letter, we have focused on the LFV three-body decay of the anti-muon, $\mu^+ \to e^+e^-e^+$. The same discussion applies to the decay of the muon, $\mu^- \to e^-e^+e^-$. In this mode, the transverse polarizations of the electrons would give $\arg[B_1C_2^* + B_2^*C_1 + 24G_1^*B_1 - 24G_2B_2^*]$ and the transverse polarizations of the emitted positrons would give $\arg[2.6B_1B_2^* - 4G_1^*C_1 - 4G_2C_2^*]$. The method of measuring the polarization described in [14] is based on studying the distribution of the photon pair from the annihilation of the emitted positron on an electron in a target. If this method is to be employed, only the polarization of the positron can be measured. Thus, to derive both combinations, the experiment has to run in both muon and anti-muon modes.

The three-body LFV decay modes of the tau lepton such as $\tau \to e\bar{e}e$ or $\tau \to \mu\bar{\mu}\mu$ can also shed light on the underlying theory. Recently it has been shown that by studying the angular distribution of the final particles in $\tau \to \mu\bar{\mu}\mu$ at the LHC, one can discriminate between various models [15]. Discussions in the present Letter also apply to the decay modes $\tau \to e\bar{e}e$ and $\tau \to \mu\bar{\mu}\mu$.

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Appendix A

In this appendix, we derive the decay rate of $\mu^+ \to e_1^+e^-e_2^+$. We first derive the decay rate into an electron of definite spin, summing over the spins of e_1^+ and e_2^+. We then concentrate on the spin of e_1^+ and derive the decay rate into a positron of definite spin, summing over the spins of the electron and the other positron.

With effective Lagrangian (3), we find

$$M\big(\mu^+ \to e_1^+e^-e_2^+\big) = B_1\bar{\mu}\frac{1+\gamma_5}{2}e_1\bar{e}\frac{1-\gamma_5}{2}e_2 - B_1\bar{\mu}\frac{1+\gamma_5}{2}e_2\bar{e}\frac{1-\gamma_5}{2}e_1 + B_2\bar{\mu}\frac{1-\gamma_5}{2}e_1\bar{e}\frac{1+\gamma_5}{2}e_2 - B_2\bar{\mu}\frac{1-\gamma_5}{2}e_2\bar{e}\frac{1+\gamma_5}{2}e_1$$

$$+C_1\bar{\mu}\frac{1+\gamma_5}{2}e_1\bar{e}\frac{1+\gamma_5}{2}e_2-C_1\bar{\mu}\frac{1+\gamma_5}{2}e_2\bar{e}\frac{1+\gamma_5}{2}e_1+C_2\bar{\mu}\frac{1-\gamma_5}{2}e_1\bar{e}\frac{1-\gamma_5}{2}e_2-C_2\bar{\mu}\frac{1-\gamma_5}{2}e_2\bar{e}\frac{1-\gamma_5}{2}e_1$$
$$-4G_1\bar{\mu}c\frac{1-\gamma_5}{2}\gamma^0e^*e_1^Tc\frac{1+\gamma_5}{2}e_2-4G_2\bar{\mu}c\frac{1+\gamma_5}{2}\gamma^0e^*e_1^Tc\frac{1-\gamma_5}{2}e_2. \tag{20}$$

A.1. Decay rate into e^- with a given spin

Consider the decay $\mu^+\to e_1^+e^-e_2^+$ in the rest frame of the muon. Since we are interested in the spin of the electron, it is convenient to use the coordinate system defined as $\hat{T}_3^-\equiv\vec{P}_{e^-}/|\vec{P}_{e^-}|$, $\hat{T}_2^-\equiv(\hat{T}_3^-\times\vec{s}_\mu)/|\hat{T}_3^-\times\vec{s}_\mu|$ and $\hat{T}_1^-\equiv\hat{T}_2^-\times\hat{T}_3^-$. In this coordinate system,

$$P_{\mu^+}=(m_\mu,0,0,0),\qquad P_{e_1^+}=E_{e_1^+}(1,\sin\alpha\cos\phi_+,\sin\alpha\sin\phi_+,\cos\alpha),$$
$$P_{e^-}=(E_e,0,0,E_e),\qquad P_{e_2^+}=(m_\mu-E_{e_1^+}-E_e,-E_{e_1^+}\sin\alpha\cos\phi_+,-E_{e_1^+}\sin\alpha\sin\phi_+,-E_e-E_{e_1^+}\cos\alpha), \tag{21}$$

where the electron mass is neglected (see Fig. 1(a)). Writing the kinematics and neglecting effects of $O(m_e^2/m_\mu^2)\ll1$, we find

$$\cos\alpha=\frac{m_\mu^2-2m_\mu E_{e_1^+}-2m_\mu E_e+2E_{e_1^+}E_e}{2E_{e_1^+}E_e}. \tag{22}$$

Summing over the spins of the emitted positrons, we find that

$$(2\pi)^4\int_0^{2\pi}(2E_{e_1^+})(2E_{e_2^+})\sum_{\vec{s}_{e_1^+},\vec{s}_{e_2^+}}|M|^2\,d\phi_+$$
$$=\big(|B_1|^2|c_{e^-}|^2+|B_2|^2|d_{e^-}|^2\big)E_{e_1^+}\big[\big(m_\mu-E_{e_1^+}(1-\cos\alpha)\big)+(1-\cos\alpha)(m_\mu-E_e-E_{e_1^+})\big]$$
$$+\big(|B_1|^2|c_{e^-}|^2-|B_2|^2|d_{e^-}|^2\big)E_{e_1^+}\big[\cos\alpha\big[m_\mu-E_{e_1^+}(1-\cos\alpha)\big]-(1-\cos\alpha)(E_{e_1^+}\cos\alpha+E_e)\big]\mathbb{P}_\mu\cos\theta_-$$
$$+\big[\big(|C_1|^2+16|G_2|^2\big)|d_{e^-}|^2(1+\mathbb{P}_\mu\cos\theta_-)+\big(|C_2|^2+16|G_1|^2\big)|c_{e^-}|^2(1-\mathbb{P}_\mu\cos\theta_-)\big]E_{e_1^+}\big[m_\mu-E_e(1-\cos\alpha)\big]$$
$$-2\mathbb{P}_\mu\,\mathrm{Re}\big[B_1B_2^*d_{e^-}c_{e^-}^*\big]E_{e_1^+}(1-\cos\alpha)\big(E_{e_1^+}(1-\cos\alpha)-m_\mu\big)\sin\theta_-$$
$$+8\mathbb{P}_\mu\big(\mathrm{Re}\big[G_1C_1^*d_{e^-}c_{e^-}^*\big]+\mathrm{Re}\big[G_2C_2^*d_{e^-}^*c_{e^-}\big]\big)E_{e_1^+}\big(E_e(1-\cos\alpha)-m_\mu\big)\sin\theta_-, \tag{23}$$

where $(c_{e^-}d_{e^-})$ determines the spin of the emitted electron [see Eq. (8)].

The differential rate of the decay into an electron in a direction that makes an angle of θ_- with the spin of the initial muon is

$$\sum_{\vec{s}_{e_1^+},\vec{s}_{e_2^+}}\frac{d\Gamma(\mu^+\to e_1^+e^-e_2^+)}{d\Omega}=\int_0^{2\pi}\int_0^{m_\mu/2}\int_{m_\mu/2-E_e}^{m_\mu/2}\sum_{\vec{s}_{e_1^+},\vec{s}_{e_2^+}}|M|^2E_eE_{e_1^+}(m_\mu-E_e-E_{e_1^+})\,dE_{e_1^+}\,dE_e\,d\phi_+,$$

where $d\Omega$ is the differential solid angle determining the orientation of the emitted electron. The factor $E_eE_{e_1^+}(m_\mu-E_e-E_{e_1^+})$ comes from the momentum-space volume for a three body decay [*i.e.*, from $\delta^4(P_\mu-P_e-P_{e_1^+}-P_{e_2^+})\,d^3P_e\,d^3P_{e_1^+}\,d^3P_{e_2^+}$]. Inserting $|M|^2$ from Eq. (23), we obtain

$$\sum_{\vec{s}_{e_1^+},\vec{s}_{e_2^+}}\frac{d\Gamma(\mu^+\to e_1^+e^-e_2^+)}{d\Omega}$$
$$=\frac{m_\mu^5}{8(2\pi)^4}\Bigg[\big(|c_{e^-}|^2|B_1|^2+|d_{e^-}|^2|B_2|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}\big(-1+3y+4x-2y^2-4x^2-4xy\big)\,dx\,dy$$
$$+\mathbb{P}_\mu\cos\theta_-\big(|B_1|^2|c_{e^-}|^2-|B_2|^2|d_{e^-}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}\bigg[1-2x-2y+2xy-\frac{2x+2y-1}{y}\big(1-2x-2y+2xy+y^2\big)\bigg]\,dx\,dy$$
$$+\big(\big(|C_1|^2+16|G_2|^2\big)|d_{e^-}|^2(1+\mathbb{P}_\mu\cos\theta_-)+\big(|C_2|^2+16|G_1|^2\big)|c_{e^-}|^2(1-\mathbb{P}_\mu\cos\theta_-)\big)\int_0^{1/2}\int_{1/2-y}^{1/2}y(1-2y)\,dx\,dy$$
$$-8\mathbb{P}_\mu\big(\mathrm{Re}\big[G_1C_1^*d_{e^-}c_{e^-}^*\big]+\mathrm{Re}\big[G_2C_2^*d_{e^-}^*c_{e^-}\big]\big)\sin\theta_-\int_0^{1/2}\int_{1/2-y}^{1/2}y(1-2y)\,dx\,dy$$
$$-2\mathbb{P}_\mu\,\mathrm{Re}\big[B_1B_2^*d_{e^-}c_{e^-}^*\big]\sin\theta_-\int_0^{1/2}\int_{1/2-y}^{1/2}(2x+2y-1)\frac{2x-1}{y}\,dx\,dy\Bigg]. \tag{24}$$

The integrals are all finite and lead to the numbers in Eq. (9)

A.2. Decay rate into e_1^+ with a given spin

Let us now concentrate on one of the positrons, e_1^+, whose momentum makes an angle of θ_+ with $\vec{s}_\mu$ (see Fig. 1(b)). To perform this analysis, it is convenient to work in the following coordinate system: $\hat{T}_3^+ \equiv \vec{P}_{e_1^+}/|\vec{P}_{e_1^+}|$, $\hat{T}_2^+ \equiv \hat{T}_3^+ \times \vec{s}_\mu/|\hat{T}_3^+ \times \vec{s}_\mu|$ and $\hat{T}_1^+ \equiv \hat{T}_2^+ \times \hat{T}_3^+$. In the rest frame of the muon and in this coordinate system,

$$\begin{aligned} &P_{\mu^+} = (m_\mu, 0, 0, 0), && P_{e^-} = E_e(1, \sin\alpha\cos\phi_-, \sin\alpha\sin\phi_-, \cos\alpha),\\ &P_{e_1^+} = (E_{e_1^+}, 0, 0, E_{e_1^+}), && P_{e_2^+} = (m_\mu - E_{e_1^+} - E_e, -E_e\sin\alpha\cos\phi_-, -E_e\sin\alpha\sin\phi_-, -E_{e_1^+} - E_e\cos\alpha), \end{aligned} \tag{25}$$

where α is given by Eq. (22).

Summing over the spins of e_2^+ and e^-, we find

$$\begin{aligned} &(2\pi)^3(2E_e)(2E_{e_2^+})\int \sum_{\vec{s}_{e^-},\vec{s}_{e_2^+}} |M|^2\, d\phi_-\\ &= \big(|B_1|^2|c_{e_1^+}|^2 + |B_2|^2|d_{e_1^+}|^2\big)E_e\big[m_\mu - E_{e_1^+}(1-\cos\alpha)\big] + \big(|B_1|^2|d_{e_1^+}|^2 + |B_2|^2|c_{e_1^+}|^2\big)(1-\cos\alpha)E_e(m_\mu - E_e - E_{e_1^+})\\ &\quad + 16\big(|G_2|^2|c_{e_1^+}|^2 + |G_1|^2|d_{e_1^+}|^2\big)E_e\big(m_\mu - E_e(1-\cos\alpha)\big) + \big(|C_1|^2|c_{e_1^+}|^2 + |C_2|^2|d_{e_1^+}|^2\big)E_e\big(m_\mu - E_e(1-\cos\alpha)\big)\\ &\quad + \mathbb{P}_\mu\big[\big(|B_1|^2|c_{e_1^+}|^2 - |B_2|^2|d_{e_1^+}|^2\big)\big[m_\mu - E_{e_1^+}(1-\cos\alpha)\big]\cos\theta_+\\ &\quad + \mathbb{P}_\mu\big(|B_2|^2|c_{e_1^+}|^2 - |B_1|^2|d_{e_1^+}|^2\big)(1-\cos\alpha)(E_e\cos\alpha + E_{e_1^+})\big]E_e\cos\theta_+\\ &\quad + \mathbb{P}_\mu\big(|B_2|^2|c_{e_1^+}|^2 - |B_1|^2|d_{e_1^+}|^2\big)\cos\phi_-\, E_e^2\sin\alpha(1-\cos\alpha)\sin\theta_+\\ &\quad + \mathbb{P}_\mu\big(|C_1|^2|c_{e_1^+}|^2 - |C_2|^2|d_{e_1^+}|^2\big)E_e\cos\alpha\big(m_\mu - E_e(1-\cos\alpha)\big)\cos\theta_+\\ &\quad + 16\mathbb{P}_\mu\big(|G_2|^2|c_{e_1^+}|^2 - |G_1|^2|d_{e_1^+}|^2\big)E_e\big(m_\mu - E_e(1-\cos\alpha)\big)\cos\alpha\cos\theta_+ + \mathbb{P}_\mu \operatorname{Re}\big[B_1C_2^* c_{e_1^+} d_{e_1^+}^*\big]m_\mu E_e\sin\theta_+(1+\cos\alpha)\\ &\quad + \mathbb{P}_\mu \operatorname{Re}\big[B_2C_1^* c_{e_1^+}^* d_{e_1^+}\big]m_\mu E_e\sin\theta_+\,(1+\cos\alpha)\\ &\quad + 4\mathbb{P}_\mu\big[\operatorname{Re}\big[G_1B_1^* c_{e_1^+}^* d_{e_1^+}\big] - \operatorname{Re}\big[G_2B_2^* d_{e_1^+}^* c_{e_1^+}\big]\big]E_e\big(m_\mu - E_e(1-\cos\alpha)\big)(1-\cos\alpha)\sin\theta_+ \end{aligned} \tag{26}$$

where $(c_{e_1^+}\ d_{e_1^+})$ determines the spin of the emitted positron [see Eq. (14)]. The differential rate of the decay into a positron in a direction that makes an angle of θ_+ with the spin of the initial muon is

$$\sum_{\vec{s}_{e^-},\vec{s}_{e_2^+}} \frac{d\Gamma(\mu^+ \to e_1^+e^-e_2^+)}{d\Omega} = \int_0^{2\pi}\int_0^{m_\mu/2}\int_{m_\mu/2-E_{e_1^+}}^{m_\mu/2} \sum_{\vec{s}_{e^-},\vec{s}_{e_2^+}} |M|^2 E_e E_{e_1^+}(m_\mu - E_e - E_{e_1^+})\, dE_e\, dE_{e_1^+}\, d\phi_-,$$

where $d\Omega$ is the differential solid angle determining the orientation of e_1^+. Inserting $|M|^2$ from Eq. (23), we obtain

$$\begin{aligned} &\sum_{\vec{s}_{e^-},\vec{s}_{e_2^+}} \frac{d\Gamma(\mu^+ \to e_1^+e^-e_2^+)}{d\Omega}\\ &= \frac{m_\mu^5}{8(2\pi)^4}\Bigg[\big(|d_{e_1^+}|^2|B_1|^2 + |c_{e_1^+}|^2|B_2|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}(1-x-y)(2x+2y-1)\,dx\,dy\\ &\quad + \big(|c_{e_1^+}|^2|B_1|^2 + |d_{e_1^+}|^2|B_2|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2} y(1-2y)\,dx\,dy + \big(|C_1|^2|c_{e_1^+}|^2 + |C_2|^2|d_{e_1^+}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2} x(1-2x)\,dx\,dy\\ &\quad + 16\big(|G_2|^2|c_{e_1^+}|^2 + |G_1|^2|d_{e_1^+}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2} x(1-2x)\,dx\,dy\\ &\quad + \mathbb{P}_\mu\cos\theta_+\big(|B_1|^2|d_{e_1^+}|^2 - |B_2|^2|c_{e_1^+}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}(2x+2y-1)\left(-y - \frac{1-2x-2y+2xy}{2y}\right)\Bigg]dx\,dy \end{aligned}$$

$$+\mathbb{P}_\mu \cos\theta_+\big(|C_1|^2|c_{e_1^+}|^2-|C_2|^2|d_{e_1^+}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}(1-2x)\frac{1-2x-2y+2xy}{2y}\,dx\,dy$$

$$+\mathbb{P}_\mu \cos\theta_+\big(|B_1|^2|c_{e_1^+}|^2-|B_2|^2|d_{e_1^+}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}y(1-2y)\,dx\,dy$$

$$+16\mathbb{P}_\mu \cos\theta_+\big(|G_2|^2|c_{e_1^+}|^2-|G_1|^2|d_{e_1^+}|^2\big)\int_0^{1/2}\int_{1/2-y}^{1/2}(1-2x)\frac{1-2x-2y+2xy}{2y}\,dx\,dy$$

$$+2\mathbb{P}_\mu\big(\mathrm{Re}\big[B_1C_2^*c_{e_1^+}d^*_{e_1^+}\big]+\mathrm{Re}\big[B_2C_1^*c^*_{e_1^+}d_{e_1^+}\big]\big)\sin\theta_+\int_0^{1/2}\int_{1/2-y}^{1/2}(2xy+0.5-x-y)\,dx\,dy$$

$$+4\mathbb{P}_\mu\big(\mathrm{Re}\big[G_1B_1^*c^*_{e_1^+}d_{e_1^+}\big]-\mathrm{Re}\big[G_2B_2^*c_{e_1^+}d^*_{e_1^+}\big]\big)\sin\theta_+\int_0^{1/2}\int_{1/2-y}^{1/2}\frac{(1-2x)(2x+2y-1)}{4x^2}\,dx\,dy. \tag{27}$$

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